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The total chromatic number of regular graphs of even order and high degree

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Abstract

The total chromatic number $\chi_T(G)$ of a graph G is the minimum number of colours needed to colour the edges and the vertices of G so that incident or adjacent elements have distinct colours. We show that if G is a regular graph of even order and $\delta(G) \geqslant \frac{2}{3}|V(G)| + \frac{23}{6}$, then $\chi_T(G) \leqslant \Delta(G) + 2$.

Keywords: Total chromatic number; Total colouring; Total colouring conjecture

1. Introduction

The graphs we shall consider are finite and simple. Let G be a graph. We denote its vertex set, edge set, complement, chromatic index, minimum degree, maximum degree, and number of components by $V(G)$, $E(G)$, $\bar{G}$, $\chi'(G)$, $\delta(G)$, $\Delta(G)$, and $C(G)$, respectively. If $F \subseteq E(G)$, then $G - F$ is the graph obtained from G by deleting F from G. If $E' \subseteq E(\bar{G})$, then $G + E'$ is the graph obtained from G by adding E'. If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S; the induced subgraph $G[V(G) \backslash S]$ is denoted by $G - S$. If $A, B \subset V(G)$ and $A \cap B = \phi$, then $G[A, B]$ denotes the bipartite subgraph of G induced by A and B, and $e_G(A, B)$ denotes the number of edges in the graph $G[A, B]$.

Given a graph G, a function $\pi : E(G) \cup V(G) \to N$ is called a (proper) *total colouring* if no two adjacent or incident elements are assigned the same colour from N. The *total*

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chromatic number of G, denoted $\chi_T(G)$, is the smallest positive integer k for which there exists a total colouring $\pi : E(G) \cup V(G) \to \{1, \ldots, k\}$.

From the definition of total chromatic number, it is clear that $\chi_T(G) \geqslant \Delta(G) + 1$. Behzad [1] and Vizing [11] independently made the following conjecture.

Total Colouring Conjecture (TCC). For any graph G, $\chi_T(G) \leqslant \Delta(G) + 2$.

This conjecture was proved for complete graphs, for graphs G having $\Delta(G) \leqslant 5$, for complete r-partite graphs, for graphs G having $\Delta(G) \geqslant \frac{3}{4}|V(G)|$, for graphs G having $\Delta(G) \geqslant |V(G)| - 5$, and planar graphs G having $\Delta(G) \neq 6$. For details, see [2,5,6,8,9,13–15].

In this paper, we show that if G is a regular graph of even order and $\delta(G) \geqslant \frac{2}{3}|V(G)| + \frac{23}{6}$, then $\chi_T(G) \leqslant \Delta(G) + 2$.

2. Useful results

We begin by stating some useful results from the literature. The first is a result of Erdös and Pósa [4].

Lemma 1. *A graph G contains a matching of size at least* $\min\{\delta(G), [\frac{1}{2}|V(G)|]\}$.

The second result is a well-known theorem of Vizing [10].

Lemma 2. *For any graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

For k and s integers, $1 \leqslant s \leqslant 4$, define the functions $g(k, s)$ and $h(s)$ as follows:

s	1	2	3	4
$g(k, s)$	$3k - 1$	$3k - 10$	$3k - 26$	$2.5k - 26$
$h(s)$	1	2	4	4

If each vertex of a graph G has degrees between k and $k + s$, G is said to be a $(k, k + s)$-graph. The third result is due to Jackson [7].

Lemma 3. *Let k, s be integers, $1 \leqslant s \leqslant 4$, and let G be a 2-connected $(k, k + s)$-graph on $n \leqslant g(k, s)$ vertices. Then G has a cycle of length at least $n - h(s)$. Furthermore, if $C(G - A) \leqslant |A|$ for all $A \subset V(G)$ with $|A| \geqslant \frac{1}{2}(n - h(s))$, then G is Hamiltonian.*

The following result is proved in [12].

Lemma 4. *Let G be a graph of even order and $\delta(G) + \Delta(G) \geqslant \frac{3}{2}|V(G)| - \frac{5}{2}$. Then $\chi_T(G) \leqslant \Delta(G) + 2$.*

Lemma 5. *Let G be a graph of order n with $\delta(G) \geqslant n/3$. If $B \subset V(G)$ and B is an independent set of G such that $|B| > (n/3) + 1$, then G is 2-connected.*

Proof. Assume that the lemma is false. Let $A = V(G) - B$. By the hypothesis of the lemma, $A \neq \phi$. Suppose that G is 1-connected. Let v be a cut vertex of G. Since $\delta(G) \geqslant n/3$, it follows that $G - \{v\}$ contains exactly two components, say G_1 and G_2. Let

$$V(G_i) = X_i \cup Y_i \quad \text{for } X_i \subset A,\ Y_i \subset B,\ i = 1, 2.$$

We may assume that $|Y_1| \leqslant |Y_2|$.

If $Y_1 = \phi$, then either $Y_2 = B$ or $Y_2 = B - \{v\}$. Since B is an independent set of G, it follows that Y_2 is an independent set of G_2. Thus, for each $u \in Y_2$, the vertex u is adjacent to at least $(n/3) - 1$ vertices of X_2 in G_2. Since $|B| > (n/3) + 1$,

$$|V(G_2)| = |X_2| + |Y_2| > \frac{n}{3} - 1 + \frac{n}{3} = \frac{2n}{3} - 1.$$

Thus,

$$|V(G_1)| = |V(G)| - |V(G_2)| - 1 < n - \left(\frac{2n}{3} - 1\right) - 1 = \frac{n}{3}.$$

However, since $\delta(G) \geqslant n/3$, this contradicts the fact that

$$|V(G_1)| \geqslant \delta(G_1) + 1 \geqslant (\delta(G) - 1) + 1 \geqslant \frac{n}{3}.$$

Thus, $Y_1 \neq \phi$. Since B is an independent set of G, it follows that Y_i is an independent set of G_i for $i = 1, 2$.

Case 1: Suppose that $v \in A$.

Since $\delta(G) \geqslant n/3$, for each $y_i \in Y_i$, the vertex y_i is adjacent to at least $(n/3) - 1$ vertices of X_i in G_i for $i = 1, 2$. Hence,

$$|X_1 \cup X_2| = |X_1| + |X_2| \geqslant \frac{2n}{3} - 2.$$

However, since $|B| > (n/3) + 1$, this contradicts the fact that

$$|X_1 \cup X_2| \leqslant |A| - 1 = |V(G)| - |B| - 1 < n - \left(\frac{n}{3} + 1\right) - 1 = \frac{2n}{3} - 2.$$

Case 2: Suppose that $v \notin A$.

Then $v \in B$. By $\delta(G) \geqslant n/3$, for each $y_i \in Y_i$, the vertex y_i is adjacent to at least $n/3$ vertices of X_i in G_i for $i = 1, 2$. Hence,

$$|X_1 \cup X_2| = |X_1| + |X_2| \geqslant \frac{2n}{3}.$$

However, since $|B| > \frac{n}{3} + 1$, this contradicts the fact that

$$|X_1 \cup X_2| \leqslant |A| = |V(G)| - |B| < n - \left(\frac{n}{3} + 1\right) = \frac{2n}{3} - 1.$$

Suppose that G is disconnected. Similarly, we can show that the lemma is valid. □

Lemma 6. *Let G be a disconnected graph and $\delta(G) \geqslant \frac{1}{3}|V(G)|$. Then G has exactly two components. Let H_1 and H_2 be the two components of G, and let $a_i \in V(H_1)$ and $b_i \in V(H_2)$, and $a_1 \neq a_2$ and $b_1 \neq b_2$ for $i=1,2$. If $H=G+\{a_1b_1, a_2b_2\}$, then H is 2-connected.*

Proof. Let $|V(G)|=n$. Assume that G has at least three components. Let G_i be a component of G for $i=1,2,3$. Since $\delta(G) \geqslant \frac{n}{3}$, $\delta(G_i) \geqslant n/3$, for $i=1,2,3$. Thus, $|V(G_i)| \geqslant (n/3)+1$ for $i=1,2,3$, and so

$$n \geqslant \sum_{i=1}^{3} |V(G_i)| \geqslant 3\left(\frac{n}{3}+1\right) = n+3,$$

which is a contradiction. Therefore, G has exactly two components.

Now we show that H is 2-connected. Assume that the result is false. By the hypothesis of the lemma,

$$\min\{\delta(H_1), \delta(H_2)\} \geqslant \frac{n}{3} \tag{1}$$

and

$$\min\{|V(H_1)|, |V(H_2)|\} \geqslant \frac{n}{3}+1. \tag{2}$$

Hence,

$$\max\{|V(H_1)|, |V(H_2)|\} \leqslant n - \left(\frac{n}{3}+1\right) \leqslant \frac{2n}{3} - 1. \tag{3}$$

From (1) and (3), and from Dirac's Theorem [3], it follows that both the graphs H_1 and H_2 are Hamiltonian. Let $C_1=a_1x_1\cdots x_ia_1$ be a Hamilton cycle in H_1, and let $C_2=b_1y_1\cdots y_jb_1$ be a Hamilton cycle in H_2. Then $P=x_1\cdots x_ia_1b_1y_1\cdots y_j$ is a Hamilton path in H. Thus, H is 1-connected. Let w be a cut vertex of H. Then the graph $H-\{w\}$ is disconnected.

Case 1: $w \in \{a_1, a_2, b_1, b_2\}$.

Without loss of generality, assume that $w=b_1$. By (1) and (3),

$$\delta(H_2-\{b_1\} \geqslant \frac{n}{3} - 1$$

and

$$|V(H_2-\{b_1\})| \leqslant \frac{2n}{3} - 2.$$

Therefore, it follows from Dirac's Theorem that the graph $H_2-\{b_1\}$ is Hamiltonian. Let $C_2'=b_2z_1\cdots z_kb_2$ be a Hamilton cycle in $H_2-\{b_1\}$. We replace $C_1=a_1x_1\cdots x_ia_1$ by $C_1=a_2x_1'\cdots x_i'a_2$. Then $P=x_1'\cdots x_i'a_2b_2z_1\cdots z_k$ is a Hamilton path in $H-\{b_1\}$. Thus, $H-\{b_1\}$ is 1-connected, which is a contradiction.

Case 2: $w \notin \{a_1, a_2, b_1, b_2\}$.

Without loss of generality, assume that $w \in H_2$. By an argument similar to that in Case 1, the graph $H_2-\{w\}$ is Hamiltonian. Let $C_2'=b_2z_1'\cdots z_k'b_2$ be a Hamilton cycle in $H_2-\{w\}$. Then $P=x_1'\cdots x_i'a_2b_2z_1'\cdots z_k'$ is a Hamilton path in $H-\{w\}$. Thus, $H-\{w\}$ is 1-connected, which is a contradiction.

Therefore, the lemma holds. □

Lemma 7. *Let H be a 2-connected graph. Form a new graph H_1 by adding a new vertex z to H and joining z to at least two vertices of $V(H)$ by edges. Then H_1 is 2-connected.*

Proof. Assume that the result is false. By the hypothesis of the lemma, H_1 is 1-connected. Let w be a cut vertex of H_1. Then the graph $H_1 - \{w\}$ is disconnected. By the hypothesis of the lemma, $w \neq z$. Since H is 2-connected, $H - \{w\}$ is 1-connected. By the hypothesis of the lemma, $\delta(H_1) \geqslant 2$, and so, $H_1 - \{w\}$ is 1-connected, which is a contradiction. □

3. The main result

Theorem 1. *Let G be a regular graph of even order and*

$$\delta(G) \geqslant \tfrac{2}{3}|V(G)| + \tfrac{23}{6}. \tag{4}$$

Then $\chi_T(G) \leqslant \Delta(G) + 2$.

Proof. Let $|V(G)| = n = 2m$. If $n \leqslant 61$, then $\frac{2}{3}|V(G)| + \frac{23}{6} \geqslant \frac{3}{4}|V(G)| - \frac{5}{4}$, and so, by the hypothesis of the theorem and Lemma 4, the result holds in this case. Thus, we assume that $n \geqslant 62$. The result is known in the case $\Delta(G) = n - 1$ (see [1]). Thus, we also assume that $\Delta(G) < n - 1$.

Let $r = n - \Delta(G) - 1$. By the hypothesis of the theorem, it follows that

$$\begin{aligned} r &= n - \Delta(G) - 1 = n - \delta(G) - 1 \\ &\leqslant n - \left(\frac{2n}{3} + \frac{23}{6}\right) - 1 = \frac{n}{3} - \frac{29}{6}. \end{aligned} \tag{5}$$

Thus, $\delta(\bar{G}) = r \leqslant \frac{1}{2}|V(G)|$. By Lemma 1, $\bar{G}$ has a matching $M = \{u_1v_1, \ldots, u_rv_r\}$ of size r. Then u_k and v_k are nonadjacent vertices in G for $k = 1, \ldots, r$.

Lemma 8. *There exist edge–disjoint matchings $M_1, \ldots, M_r$ of G such that, for $1 \leqslant k \leqslant r$, M_k misses the two vertices u_k and v_k, and does not miss any other vertex.*

Proof. We prove it by induction on k. Suppose that $k=1$. Let $G_1=G$, and let $H_1=G_1-\{u_1\}$. By (4),

$$\delta(H_1) \geqslant \frac{2n}{3} + \frac{23}{6} - 1 \geqslant \frac{1}{2}|V(H_1)|.$$

It follows from Dirac's theorem that the graph H_1 is Hamiltonian. Let C_1 be a Hamilton cycle in H_1 and let M_1 be the maximum matching in H_1 such that $M_1 \subset E(G)$ and v_1 is not incident with an edge in M_1. Therefore, G contains the matching M_1 which misses the two vertices, u_1 and v_1, and does not miss any other vertex. Thus, we can see that the lemma holds for $k = 1$. Assume that it holds for $M_1, \ldots, M_{k-1}$ and $2 \leqslant k \leqslant r$.

Let $G_k = G - (M_1 \cup \cdots \cup M_{k-1})$, and let $H_k = G_k - \{u_k\}$.

Case 1: H_k is 2-connected.

Suppose that H_k is Hamiltonian. By an argument similar to that in $k=1$, the lemma holds. Now assume that H_k is a non-Hamiltonian graph. Since G is regular, it follows that

$$\Delta(H_k) \leqslant \delta(H_k) + 2. \tag{6}$$

Using the fact that $k \leqslant r$, by (4) and (5), we have

$$\begin{aligned}\delta(H_k) &\geqslant \delta(G) - (k-1) - 1 \geqslant \delta(G) - (r-1) - 1 = \delta(G) - r \\ &\geqslant \left(\frac{2n}{3} + \frac{23}{6}\right) - \left(\frac{n}{3} - \frac{29}{6}\right) \geqslant \frac{n}{3} + \frac{26}{3},\end{aligned} \tag{7}$$

i.e., $|V(H_k)| \leqslant n \leqslant 3\delta(H_k) - 26 \leqslant 3\delta(H_k) - 10$. By Lemma 3, there exists a subset A of $V(H_k)$ such that

$$|A| \geqslant \tfrac{1}{2}(|V(H_k)| - 2) = \tfrac{1}{2}((n-1) - 2) = m - \tfrac{3}{2} \quad \text{and} \quad C(H_k - A) > |A|.$$

Thus,

$$|A| = m - 1 \quad \text{and} \quad C(H_k - A) = m. \tag{8}$$

Let $B = V(H_k) - A$. By (8), B is an independent set of H_k, and

$$|B| = |V(H_k) - A| = m. \tag{9}$$

By (4) and (9),

$$\delta(G[B]) \geqslant \delta(G) - |B| \geqslant \frac{2n}{3} + \frac{23}{6} - \frac{n}{2} = \frac{n}{6} + \frac{23}{6}.$$

Thus,

$$|E(G[B])| \geqslant \frac{1}{2}\delta(G[B])|B| \geqslant \frac{1}{2}\left(\frac{n}{6} + \frac{23}{6}\right)\frac{n}{2} \geqslant \frac{n^2}{24}.$$

Using the fact that $k \leqslant r$, by (5), we have

$$\frac{|E(G[B])|}{k-1} \geqslant \frac{\frac{n^2}{24}}{r-1} \geqslant \frac{\frac{n^2}{24}}{(\frac{n}{3} - \frac{29}{6}) - 1} > \frac{n}{8}.$$

Since $n \geqslant 62$, at least one of $\{M_1, \ldots, M_{k-1}\}$, say, M_{k-1}, contains at least seven disjoint edges $b_1b_2, \ldots, b_{13}b_{14}$ in G, which are only incident with the vertices of B (in G). Since $|B| = |A| + 1$, it follows that

$$|M_{k-1} \cap E(G[A])| \geqslant 4.$$

Thus, we assume that $a_1a_2, a_3a_4, a_5a_6, a_7a_8 \in M_{k-1} \cap E(G[A])$.Then u_{k-1} and v_{k-1} are not incident with any edge of $\{a_1a_2, a_3a_4, a_5a_6, a_7a_8, b_1b_2, \ldots, b_{13}b_{14}\}$ in G. Let $G'_{k-1} = (G_k + M_{k-1}) - \{a_1a_2, b_1b_2\}$ and let $H'_{k-1} = G'_{k-1} - \{u_{k-1}\}$. By an argument similar to that in (6) and (7), we have

$$\Delta(H'_{k-1}) \leqslant \delta(H'_{k-1}) + 3 \tag{10}$$

and

$$\delta(H'_{k-1}) \geqslant \delta(G) - (r-1) - 1 = \delta(G) - r \geqslant \frac{n}{3} + \frac{26}{3}. \tag{11}$$

By (7),

$$\delta(H_k - \{u_{k-1}\}) \geqslant \frac{n}{3} + \frac{23}{3}.$$

Since $|B - \{u_{k-1}\}| \geqslant (n/2) - 1$ and $n \geqslant 62$, $|B - \{u_{k-1}\}| > (n/3) + 1$. Since B is an independent set of H_k, it follows from Lemma 5 that $H_k - \{u_{k-1}\}$ is 2-connected. Since $H_k - \{u_{k-1}\}$ is a subgraph of H'_{k-1}, it follows from (11) and Lemma 7 that H'_{k-1} is also 2-connected. We shall show that H'_{k-1} is Hamiltonian. Assume that the result is false. By (10), (11), and Lemma 3, there exists a subset A' of $V(H'_{k-1})$ such that

$$|A'| \geqslant \tfrac{1}{2}(|V(H'_{k-1})| - 4) = \tfrac{1}{2}((n-1) - 4) = m - \tfrac{5}{2} \quad \text{and} \quad C(H'_{k-1} - A') > |A'|.$$

Hence,

$$m - 2 \leqslant |A'| \leqslant m - 1 \quad \text{and} \quad m - 1 \leqslant C(H'_{k-1} - A') \leqslant m + 1. \tag{12}$$

Case 1.1: $|A' \cap B| \geqslant (m/2) - 1$.

Suppose that $(B - \{u_{k-1}\}) - A' = \phi$. Then $B - \{u_{k-1}\} \subseteq A'$. By (9) and (12), $B - \{u_{k-1}\} = A'$. Thus, $A \subset V(H'_{k-1} - A')$. Therefore, we have $a_3a_4, a_5a_6, a_7a_8 \in E(H'_{k-1} - A')$, and so

$$C(H'_{k-1} - A') \leqslant |V(H'_{k-1})| - |A'| - 3 \leqslant (2m-1) - (m-2) - 3 \leqslant m - 2,$$

which contradicts (12). Thus,

$$(B - \{u_{k-1}\}) - A' \neq \phi.$$

Either $A' = (A' \cap A) \cup (A' \cap B) \cup \{u_k\}$ or $A' = (A' \cap A) \cup (A' \cap B)$, and therefore we have

$$|A' \cap A| \leqslant |A'| - |A' \cap B| \leqslant (m-1) - \left(\frac{m}{2} - 1\right) = \frac{m}{2}.$$

By (7),

$$\begin{aligned}\delta(H_k) - |(A' \cap A) \cup \{u_{k-1}\}| &\geqslant \frac{n}{3} + \frac{26}{3} - \frac{m}{2} - 1 \\ &\geqslant \frac{m}{6} + \frac{23}{3} \geqslant 7.\end{aligned}$$

Since B is an independent set of H_k and since $H_k - \{u_{k-1}\}$ is a subgraph of H'_{k-1}, there exists a vertex $u \in (B - \{u_{k-1}\}) - A'$ such that

$$e_{H'_{k-1}}(\{u\}, A - A' \cup \{u_{k-1}\}) \geqslant e_{H'_{k-1}}(\{u\}, A - (A' \cap A) \cup \{u_{k-1}\}) \geqslant 7.$$

Then $\Delta(H'_{k-1} - A') \geqslant 7$. Thus,

$$C(H'_{k-1} - A') \leqslant |V(H'_{k-1})| - |A'| - 7 \leqslant (2m-1) - (m-2) - 7 \leqslant m - 2,$$

which contradicts (12).

Case 1.2: $|A' \cap A| \geqslant (m/2) - 1$.

Case 1.2.1: Suppose that $|A - A'| = 0, 1, 2$. By (9),

$$|B - A' \cup \{u_{k-1}\}| \geqslant m - 3.$$

Thus, there exist at most three vertices of $\{b_3, \ldots, b_{14}\}$, say, b_3, b_5, b_7, such that $b_3, b_5, b_7 \notin B \cap (V(H'_{k-1} - A'))$. Therefore, $b_9 b_{10}, b_{11} b_{12}, b_{13} b_{14} \in E(H'_{k-1} - A')$, and

$$C(H'_{k-1} - A') \leqslant |V(H'_{k-1})| - |A'| - 3 \leqslant (2m - 1) - (m - 2) - 3 \leqslant m - 2,$$

which contradicts (12).

Case 1.2.2: Suppose that $3 \leqslant |A - A'| \leqslant 11$. Since B is an independent set of H_k, it follows from (6) that H_k has at least $m\delta(H_k)$ edges which are incident with the vertices of B, at most $(m-1)(\delta(H_k) + 2)$ edges which are incident with the vertices of A. Thus, by (7), we have

$$\begin{aligned} |E(H_k[A])| &\leqslant \frac{(m-1)(\delta(H_k)+2) - m\delta(H_k)}{2} \\ &\leqslant (m-1) - \frac{1}{2}\delta(H_k) \leqslant (m-1) - \frac{1}{2}\left(\frac{n}{3} + \frac{26}{3}\right) \\ &= (m-1) - \left(\frac{m}{3} + \frac{13}{3}\right) = \frac{2m}{3} - \frac{16}{3}. \end{aligned} \tag{13}$$

Hence,

$$\delta(H_k) - |E(H_k[A])| \geqslant \frac{2m}{3} + \frac{26}{3} - \left(\frac{2m}{3} - \frac{16}{3}\right) = 14.$$

Thus, each vertex of $A - A'$ is adjacent to at least fourteen vertices of B in H_k. Since $3 \leqslant |A - A'| \leqslant 11$, there exist two vertices $x_1, x_2 \in A - A'$ such that $x_1 \neq u_{k-1}$ and $x_2 \neq u_{k-1}$. Then x_i is adjacent to at least fourteen vertices of B in H_k for $i = 1, 2$. Using the fact that $|A - A'| \leqslant 11$, by (8) and (12), we have $|A' - A| \leqslant 11$. Therefore, $|A' \cap B| \leqslant 11$. Thus, x_i is adjacent to at least two vertices of $B - A'$ in $H_k - \{u_{k-1}\}$ for $i = 1, 2$. Since $H_k - \{u_{k-1}\}$ is a subgraph of H'_{k-1}, it follows that

$$C(H'_{k-1} - A') \leqslant |V(H'_{k-1})| - |A'| - 3 \leqslant (2m - 1) - (m - 2) - 3 \leqslant m - 2,$$

which contradicts (12).

Case 1.2.3: Suppose that $|A - A'| \geqslant 12$. Let $p = |A - A'|$. Either $u_{k-1} \in A - A'$ or $u_{k-1} \notin A - A'$, and therefore, by (7) and (13), $A - A'$ contains at least a vertex $v \neq u_{k-1}$

such that

$$\begin{aligned}e_{H_k}(\{v\}, B) &\geqslant \frac{(p-1)\left(\frac{n}{3}+\frac{26}{3}\right)-2\left(\frac{2m}{3}-\frac{16}{3}\right)}{p}\\ &\geqslant \frac{(p-1)}{p}\left(\frac{2m}{3}+\frac{26}{3}\right)-\frac{4m}{3p}+\frac{32}{3p}\\ &\geqslant \frac{11}{12}\left(\frac{2m}{3}+\frac{26}{3}\right)-\frac{4}{3}\frac{m}{12}\\ &=\frac{m}{2}+\frac{11}{12}\frac{26}{3}\geqslant \frac{m}{2}+7. \end{aligned} \tag{14}$$

Thus,

$$e_{H_k-\{u_{k-1}\}}(\{v\}, B)\geqslant \frac{m}{2}+6. \tag{15}$$

Using (12), and the fact that $|A'\cap A|\geqslant (m/2)-1$, yields

$$|A'\cap B|\leqslant |A'|-|A'\cap A|\leqslant (m-1)-\left(\frac{m}{2}-1\right)=\frac{m}{2}. \tag{16}$$

Since $H_k-\{u_{k-1}\}$ is a subgraph of H'_{k-1}, it follows from (15) and (16) that

$$\begin{aligned}e_{H'_{k-1}}(\{v\}, B-(A'\cup\{u_{k-1}\})\geqslant e_{H'_{k-1}}(\{v\}, B-(A'\cap B)\cup\{u_{k-1}\})\\ \geqslant \left(\frac{m}{2}+6\right)-\left(\frac{m}{2}+1\right)\geqslant 5.\end{aligned}$$

Hence,

$$\Delta(H'_{k-1}-A')\geqslant 5.$$

Thus,

$$C(H'_{k-1}-A')\leqslant |V(H'_{k-1})|-|A'|-5\leqslant (2m-1)-(m-2)-5\leqslant m-2,$$

which contradicts (12).

Thus, the graph H'_{k-1} is Hamiltonian. Let C'_{k-1} be a Hamilton cycle in H'_{k-1} and let M'_{k-1} be the maximum matching in H'_{k-1} such that $M'_{k-1}\subset E(C'_{k-1})$ and v_{k-1} is not incident with an edge in M'_{k-1}. Therefore, G contains the matching M'_{k-1} which misses the two vertices u_{k-1} and v_{k-1}, and does not miss any other vertex.

Let $G'_k=G-(M_1\cup\cdots\cup M_{k-2}\cup M'_{k-1})$ and let $H'_k=G'_k-\{u_k\}$. By an argument similar to that in (6) and (7), we have

$$\Delta(H'_k)\leqslant \delta(H'_k)+2 \tag{17}$$

and

$$\delta(H'_k)\geqslant \frac{n}{3}+\frac{26}{3}. \tag{18}$$

Since $|B|\geqslant \frac{n}{2}$ and $n\geqslant 62$, $|B|>\frac{n}{3}+1$. Since B is an independent set of H_k, it follows from (7) and Lemma 5 that the graph $H_k-M'_{k-1}$ is 2-connected. Since $H_k-M'_{k-1}$ is a spanning subgraph of H_k', it follows that H_k' is also 2-connected. We shall show that H_k' is Hamiltonian. Assume that the result is false. By (17), (18), and Lemma 3, there exists a subset A'' of $V(H_k')$ such that

$$|A''| \geqslant \tfrac{1}{2}(|V(H_k')| - 2) = \tfrac{1}{2}((2m-1)-2) = m - \tfrac{3}{2} \quad \text{and} \quad C(H_k' - A'') > |A''|.$$

Thus,

$$|A''| = m - 1 \quad \text{and} \quad C(H_{k-1}' - A'') = m. \tag{19}$$

Suppose that $A - A'' = \phi$. By (8) and (19), $A'' = A$. Since $b_1b_2 \in H_k'[B]$, it follows that $b_1b_2 \in E(H_k' - A'')$. Then

$$C(H_k' - A'') \leqslant |V(H_k')| - |A''| - 1 = (2m-1) - (m-1) - 1 < m,$$

which contradicts (19). Thus, $A - A'' \neq \phi$. Clearly, $B - A'' \neq \phi$.

Case 1.1′: $|A'' \cap B| \geqslant m/2$.

Clearly, $|A'' \cap A| \leqslant (m/2) - 1$. Since B is an independent set of H_k, it follows that $\Delta(H_k'[B]) \leqslant 1$. By (18),

$$\begin{aligned} e_{H_k'}(\{b\}, A - A'') &\geqslant \frac{n}{3} + \frac{26}{3} - |A'' \cap A| - 1 \\ &\geqslant \frac{2m}{3} + \frac{26}{3} - \left(\frac{m}{2} - 1\right) - 1 \\ &= \frac{m}{6} + \frac{26}{3} \quad \text{for all } b \in B - A''. \end{aligned}$$

Since $B - A'' \neq \phi$, $\Delta(H_k' - A'') \geqslant (m/6) + \frac{26}{3}$ and so

$$\begin{aligned} C(H_k' - A'') &\leqslant |V(H_k')| - |A''| - \left(\frac{m}{6} + \frac{26}{3}\right) = (2m-1) - (m-1) - \left(\frac{m}{6} + \frac{26}{3}\right) \\ &\leqslant m - 1, \end{aligned}$$

which contradicts (19).

Case 1.2′: $|A'' \cap A| \geqslant m/2$.

Suppose that $1 \leqslant |A - A''| \leqslant 7$. By (13),

$$|E((H_k - M_{k-1}')[A])| \leqslant \left(\frac{2m}{3} - \frac{16}{3}\right).$$

By (7),

$$\begin{aligned} e_{H_k - M_{k-1}'}(\{v\}, B) &\geqslant \delta(H_k) - 1 - \left(\frac{2m}{3} - \frac{16}{3}\right) \\ &\geqslant \frac{2m}{3} + \frac{26}{3} - 1 - \left(\frac{2m}{3} - \frac{16}{3}\right) = 13 \quad \text{for each } v \in A. \end{aligned}$$

Since $H_k - M_{k-1}'$ is a subgraph of H_k', it follows that

$$e_{H_k'}(\{v\}, B) \geqslant 13 \quad \text{for each } v \in A.$$

Using the fact that $|A - A''| \leqslant 7$, by (8) and (19), we have $|A'' - A| \leqslant 7$. Therefore, $|B \cap A''| \leqslant 7$. Thus,

$$e_{H_k'}(\{v\}, B - A'') = e_{H_k'}(\{v\}, B - B \cap A'') \geqslant 13 - |B \cap A''| \geqslant 6 \quad \text{for each } v \in A.$$

Since $|A - A''| \geqslant 1$, there exists a vertex $v \in A - A''$ such that

$$e_{H_k'}(\{v\}, B - A'') \geqslant 6.$$

Hence,

$$C(H_k' - A'') \leqslant |V(H_k')| - |A''| - 6 = (2m - 1) - (m - 1) - 6 \leqslant m - 1,$$

which contradicts (19).

Suppose that $|A - A''| \geqslant 8$. Let $q = |A - A''|$. Since $H_k - M_{k-1}'$ is a subgraph of H_k', by an argument similar to that in (14), there exists a vertex $w \in A - A''$ such that

$$\begin{aligned} e_{H_k'}(\{w\}, B) \geqslant e_{H_k - M_{k-1}'}(\{w\}, B) &\geqslant \frac{q\left(\frac{2m}{3} + \frac{26}{3} - 1\right) - 2\left(\frac{2m}{3} - \frac{16}{3}\right)}{q} \\ &\geqslant \frac{2m}{3} + \frac{23}{3} - \frac{4m}{3q} + \frac{32}{3q} \\ &\geqslant \frac{2m}{3} + \frac{23}{3} - \frac{4}{3}\frac{m}{8} \geqslant \frac{m}{2} + \frac{23}{3}. \end{aligned}$$

Since $|A'' \cap A| \geqslant m/2$,

$$|A'' \cap B| \leqslant |A''| - |A'' \cap A| \leqslant (m - 1) - \frac{m}{2} = \frac{m}{2} - 1.$$

Hence,

$$e_{H_k'}(\{w\}, B - A'') = e_{H_k'}(\{w\}, B - A'' \cap B) \geqslant \left(\frac{m}{2} + \frac{23}{3}\right) - \left(\frac{m}{2} - 1\right) > 8,$$

and so

$$C(H_k' - A'') \leqslant |V(H_k')| - |A''| - 8 = (2m - 1) - (m - 1) - 8 \leqslant m - 1,$$

which contradicts (19).

Thus, the graph, H_k' is Hamiltonian. Let C_k' be a Hamilton cycle in H_k' and let M_k be the maximum matching in H_k' such that $M_k \subset E(C_k')$ and v_k is not incident with an edge in M_k. Then we replace M_{k-1} by M_{k-1}' in $\{M_1, \ldots, M_{k-1}, M_k\}$ and still denote M_{k-1}' by M_{k-1} in $\{M_1, \ldots, M_{k-1}', M_k\}$. Therefore, G contains the matching M_k which misses the two vertices u_k and v_k, and does not miss any other vertex.

Case 2: H_k is 1-connected, but not 2-connected.

Let z be a cut vertex of H_k. By (7),

$$\delta(H_k - \{z\}) \geqslant \frac{n}{3} + \frac{23}{3}. \tag{20}$$

By Lemma 6, $H_k - \{z\}$ has exactly two components, say $H_k^{(1)}$ and $H_k^{(2)}$. From (20),

$$\min\{\delta(H_k^{(1)}), \delta(H_k^{(2)})\} \geqslant \frac{n}{3} + \frac{23}{3} \tag{21}$$

and

$$\min\{|V(H_k^{(1)})|, |V(H_k^{(2)})|\} \geqslant \frac{n}{3} + \frac{26}{3}. \tag{22}$$

Hence,

$$\max\{|V(H_k^{(1)})|, |V(H_k^{(2)})|\} \leqslant n - \left(\frac{n}{3} + \frac{26}{3}\right) \leqslant \frac{2n}{3} - \frac{26}{3}. \tag{23}$$

We may assume that $|V(H_k^{(1)})| \leqslant |V(H_k^{(2)})|$. Then

$$|V(H_k^{(1)})| \leqslant \frac{n}{2} - 1. \tag{24}$$

Case 2.1: Suppose that $z \neq v_k$.

We may assume that $v_k \in V(H_k^{(2)})$. By (21) and (23), and Dirac's Theorem, it follows that both the graphs $H_k^{(1)}$ and $H_k^{(2)} - \{v_k\}$ are Hamiltonian. Let $C_k^{(1)} = x_0 x_1 \cdots x_i x_0$ be a Hamilton cycle in $H_k^{(1)}$, and let $C_k^{(2)} = y_0 y_1 \cdots y_j y_0$ be a Hamilton cycle in $H_k^{(2)} - \{v_k\}$. We may assume that z is adjacent to both the vertices x_0 and y_0 in H_k. Then $P = x_1 \cdots x_i x_0 z y_0 y_1 \cdots y_j$ is a Hamilton path in $H_k - \{v_k\}$. Let M_k be the maximum matching in $H_k - \{v_k\}$ such that $M_k \subset E(P)$. Thus, G contains the matching M_k which misses the two vertices u_k and v_k, and does not miss any other vertex.

Case 2.2: Suppose that $z = v_k$.

By (4), (22), and (24),

$$\begin{aligned}
e_G(V(H_k^{(1)}), V(H_k^{(2)})) &\geqslant |V(H_k^{(1)})|(\delta(G) - 2 - \Delta(G[V(H_k^{(1)})])) \\
&\geqslant |V(H_k^{(1)})|(\delta(G) - 2 - (|V(H_k^{(1)})| - 1)) \\
&\geqslant \left(\frac{n}{3} + \frac{26}{3}\right)\left(\frac{2n}{3} + \frac{23}{6} - 2 - \left(\frac{n}{2} - 1 - 1\right)\right) \\
&\geqslant \left(\frac{n}{3} + \frac{26}{3}\right)\left(\frac{n}{6} + \frac{23}{6}\right) \geqslant \frac{n^2}{18}.
\end{aligned}$$

Using the fact that $k \leqslant r$ and $n \geqslant 62$, by (5), we have

$$\frac{e_G(V(H_k^{(1)}), V(H_k^{(2)}))}{k-1} \geqslant \frac{\frac{n^2}{18}}{r} \geqslant \frac{\frac{n^2}{18}}{\frac{n}{3} - \frac{29}{6}} \geqslant \frac{n}{6} > 7.$$

Thus, at least one of $\{M_1, \ldots, M_{k-1}\}$, say, M_{k-1}, contains at least seven disjoint edges in $(G - \{u_{k-1}, v_{k-1}, u_k, v_k\} - (E(G[V(H_k^{(1)})]) \cup E(G[V(H_k^{(2)})]))$. Thus, we assume that $a_i b_i \in M_{k-1}$ such that $a_i \in V(H_k^{(1)})$ and $b_i \in V(H_k^{(2)})$ for $i = 1, 2, 3, 4$, and $a_i, b_i \notin \{u_{k-1}, v_{k-1}, u_k, v_k\}$ (for $i = 1, 2, 3, 4$).

Let $G''_{k-1}=(G_k+M_{k-1})-\{a_1b_1, a_2b_2\}$ and let $H''_{k-1}=G''_{k-1}-\{u_{k-1}\}$. By (7), $\delta(H_k-\{u_{k-1}, v_k\})>n/3$, so the graph $(H_k-\{u_{k-1}, v_k\})+\{a_3b_3, a_4b_4\}$ satisfies the conditions of Lemma 6, and therefore it is 2-connected. Using (7) and the fact that $\delta(G_k)\geqslant\delta(H_k)$, we have $\delta(G_k-\{u_{k-1}\})\geqslant 7$, and so, $d_{G_k-\{u_{k-1}\}}(u_k)\geqslant 7$ and $d_{G_k-\{u_{k-1}\}}(v_k)\geqslant 7$. Since $(H_k-\{u_{k-1}, v_k\})+\{a_3b_3, a_4b_4\}$ is 2-connected, it follows from Lemma 7 that the graph $(G_k-\{u_{k-1}\})+\{a_3b_3, a_4b_4\}$ is also 2-connected. Since $(G_k-\{u_{k-1}\})+\{a_3b_3, a_4b_4\}$ is a spanning subgraph of H''_{k-1}, H''_{k-1} is 2-connected. We shall show that H''_{k-1} is Hamiltonian. Assume that the result is false. By an argument similar to that in (10) and (11), we have

$$\Delta(H''_{k-1})\leqslant\delta(H''_{k-1})+3$$

and

$$\delta(H''_{k-1})\geqslant\frac{n}{3}+\frac{26}{3}.$$

By Lemma 3, there exists a subset S of $V(H''_{k-1})$ such that

$$|S|\geqslant\tfrac{1}{2}(|V(H''_{k-1})|-4)=\tfrac{1}{2}((n-1)-4)=m-\tfrac{5}{2}\quad\text{and}\quad C(H''_{k-1}-S)>|S|.$$

Hence,

$$m-2\leqslant|S|\leqslant m-1\quad\text{and}\quad m-1\leqslant C(H''_{k-1}-S)\leqslant m+1. \tag{25}$$

Let $S_1=S\cap V(H_k^{(1)})$ and let $S_2=S\cap V(H_k^{(2)})$. We may assume that $|S_1|\leqslant|S_2|$. Then $|S_1|\leqslant(m/2)-1$. By (21),

$$\delta(H_k^{(1)})-|S_1|\geqslant\frac{n}{3}+\frac{23}{3}-\left(\frac{m}{2}-1\right)=\frac{m}{6}+\frac{26}{3}.$$

Either $u_{k-1}\in V(H_k^{(1)})$ or $u_{k-1}\notin V(H_k^{(1)})$, and therefore we have

$$\Delta(H_k^{(1)}-S)\geqslant\frac{m}{6}+\frac{26}{3}-1=\frac{m}{6}+\frac{23}{3}.$$

Hence,

$$\begin{aligned}C(H''_{k-1}-S)&\leqslant|V(H''_{k-1})|-|S|-\left(\frac{m}{6}+\frac{23}{3}\right)\\&\leqslant(2m-1)-(m-2)-\left(\frac{m}{6}+\frac{23}{3}\right)\leqslant m-2,\end{aligned}$$

which contradicts (25).

Thus, the H''_{k-1} is Hamiltonian. Let C''_{k-1} be a Hamilton cycle in H''_{k-1} and let M''_{k-1} be the maximum matching in H''_{k-1} such that $M''_{k-1}\subset E(C''_{k-1})$ and v_{k-1} is not incident with an edge in M''_{k-1}. Therefore, G contains the matching M''_{k-1} which misses the two vertices u_{k-1} and v_{k-1}, and does not miss any other vertex.

Let $G''_k=G-(M_1\cup\cdots\cup M_{k-2}\cup M''_{k-1})$, and let $H''_k=G''_k-\{u_k\}$. By (7), $\delta((H_k-\{v_k\})-M''_{k-1})>\frac{n}{3}$. Thus, the graph $((H_k-\{v_k\})-M''_{k-1})+\{a_1b_1, a_2b_2\}$ satisfies the conditions of Lemma 6, and so it is 2-connected. Let $G_k^{(1)}=(H_k-M''_{k-1})+\{a_1b_1, a_2b_2\}$. By (7),

$d_{G_K^{(1)}}(v_k) \geqslant 7$. Since $((H_k - \{v_k\}) - M''_{k-1}) + \{a_1b_1, a_2b_2\}$ is 2-connected, it follows from Lemma 7 that the graph $G_k^{(1)}$ is also 2-connected. Since $G_k^{(1)}$ is a spanning subgraph of H''_k, H''_k is 2-connected. Then as in Case 2.2, H''_k is also Hamiltonian. Let C_k be a Hamilton cycle in H''_k and let M_k be the maximum matching in H''_k such that $M_k \subset E(C_k)$ and v_k is not incident with an edge in M_k. Then we replace M_{k-1} by M''_{k-1} in $\{M_1, \ldots, M_{k-1}, M_k\}$ and still denote M''_{k-1} by M_{k-1} in $\{M_1, \ldots, M''_{k-1}, M_k\}$. Therefore, G contains the matching M_k which misses the two vertices u_k and v_k, and does not miss any other vertex.

Case 3: H_k is disconnected.

By (7) and Lemma 6, H_k has exactly two components, say $H_k^{(3)}$ and $H_k^{(4)}$. From (7),

$$\min\{\delta(H_k^{(3)}), \delta(H_k^{(4)})\} \geqslant \frac{n}{3} + \frac{26}{3} \tag{26}$$

and

$$\min\{|V(H_k^{(3)})|, |V(H_k^{(4)})|\} \geqslant \frac{n}{3} + \frac{29}{3}. \tag{27}$$

Hence,

$$\max\{|V(H_k^{(3)})|, |V(H_k^{(4)})|\} \leqslant n - \left(\frac{n}{3} + \frac{29}{3}\right) \leqslant \frac{2n}{3} - \frac{29}{3}. \tag{28}$$

We may assume that $|V(H_k^{(3)})| \leqslant |V(H_k^{(4)})|$. Then

$$|V(H_k^{(3)})| \leqslant \frac{n}{2} - 1. \tag{29}$$

By (4), (27), and (29),

$$\begin{aligned} e_G(V(H_k^{(3)}), V(H_k^{(4)})) &\geqslant |V(H_k^{(3)})|(\delta(G) - 1 - \Delta(G[V(H_k^{(3)})])) \\ &\geqslant |V(H_k^{(3)})|(\delta(G) - 1 - (|V(H_k^{(3)})| - 1)) \\ &\geqslant \left(\frac{n}{3} + \frac{29}{3}\right)\left(\frac{2n}{3} + \frac{23}{6} - 1 - \left(\frac{n}{2} - 1 - 1\right)\right) \\ &\geqslant \left(\frac{n}{3} + \frac{29}{3}\right)\left(\frac{n}{6} + \frac{26}{6}\right) \geqslant \frac{n^2}{18}. \end{aligned}$$

Then as in Case 2.2, we have

$$\frac{e_G(V(H_k^{(3)}), V(H_k^{(4)}))}{k-1} > 7.$$

Thus, at least one of $\{M_1, \ldots, M_{k-1}\}$, say, M_{k-1}, contains at least six disjoint edges in $(G - \{u_{k-1}, v_{k-1}, u_k, v_k\} - (E(G[V(H_k^{(3)})]) \cup E(G[V(H_k^{(4)})]))$. Thus, we assume that $a_ib_i \in M_{k-1}$ such that $a_i \in V(H_k^{(3)})$ and $b_i \in V(H_k^{(4)})$ for $i = 1, 2, 3, 4$, and $a_i, b_i \notin \{u_{k-1}, v_{k-1}, u_k, v_k\}$ (for $i = 1, 2, 3, 4$).

Let $G'''_{k-1} = (G_k + M_{k-1}) - \{a_1b_1, a_2b_2\}$ and let $H'''_{k-1} = G'''_{k-1} - \{u_{k-1}\}$. By (7), the graph $(H_k - \{u_{k-1}\}) + \{a_3a_4, b_3b_4\}$ satisfies the conditions of Lemma 6, and so it is

2-connected. Using (7) and the fact that $\delta(G_k)\geqslant\delta(H_k)$, we have $\delta(G_k-\{u_{k-1}\})\geqslant 7$, and so $d_{G_k-\{u_{k-1}\}}(u_k)\geqslant 7$. Since $(H_k-\{u_{k-1}\})+\{a_3b_3, a_4b_4\}$ is 2-connected, it follows from Lemma 7 that the graph $(G_k-\{u_{k-1}\})+\{a_3b_3, a_4b_4\}$ is also 2-connected. Since $(G_k-\{u_{k-1}\})+\{a_3b_3, a_4b_4\}$is a spanning subgraph of H'''_{k-1}, H'''_{k-1} is 2-connected. We shall show that H'''_{k-1} is Hamiltonian. Assume that the result is false. By an argument similar to that in (10) and (11), we have

$$\Delta(H'''_{k-1})\leqslant\delta(H'''_{k-1})+3$$

and

$$\delta(H'''_{k-1})\geqslant\frac{n}{3}+\frac{26}{3}.$$

By Lemma 3, there exists a subset S' of $V(H'''_{k-1})$ such that

$$|S'|\geqslant\tfrac{1}{2}(|V(H'''_{k-1})|-4)=\tfrac{1}{2}((n-1)-4)=m-\tfrac{5}{2}\quad\text{and}\quad C(H'''_{k-1}-S')>|S'|.$$

Hence,

$$m-2\leqslant|S'|\leqslant m-1\quad\text{and}\quad m-1\leqslant C(H'''_{k-1}-S')\leqslant m+1. \tag{30}$$

Let $S'_1=S'\cap V(H_k^{(3)})$ and let $S'_2=S'\cap V(H_k^{(4)})$. We may assume that $|S'_1|\leqslant|S'_2|$. Then $|S'_1|\leqslant(m/2)-1$. By (26),

$$\delta(H_k^{(3)})-\left|S'_1\right|\geqslant\frac{n}{3}+\frac{26}{3}-\left(\frac{m}{2}-1\right)=\frac{m}{6}+\frac{29}{3}.$$

Either $u_{k-1}\in V(H_k^{(3)})$ or $u_{k-1}\notin V(H_k^{(3)})$, and therefore we have

$$\Delta(H_k^{(3)}-S')\geqslant\frac{m}{6}+\frac{29}{3}-1=\frac{m}{6}+\frac{26}{3}.$$

Hence,

$$\begin{aligned}C(H'''_{k-1}-S')&\leqslant|V(H'''_{k-1})|-|S'|-\left(\frac{m}{6}+\frac{26}{3}\right)\\&\leqslant(2m-1)-(m-2)-\left(\frac{m}{6}+\frac{26}{3}\right)\leqslant m-2,\end{aligned}$$

which contradicts (30).

Thus, the graph H'''_{k-1} is Hamiltonian. Let C'''_{k-1} be a Hamilton cycle in H'''_{k-1} and let M'''_{k-1} be the maximum matching in H'''_{k-1} such that $M'''_{k-1}\subset E(C'''_{k-1})$ and v_{k-1} is not incident with an edge in M'''_{k-1}. Therefore, G contains the matching M'''_{k-1} which misses the two vertices u_{k-1} and v_{k-1}, and does not miss any other vertex.

Let $G'''_k=G-(M_1\cup\cdots\cup M_{k-2}\cup M'''_{k-1})$, and let $H'''_k=G'''_k-\{u_k\}$. By (7), $\delta(H_k-M'''_{k-1})>n/3$. Thus, $(H_k-M'''_{k-1})+\{a_1b_1, a_2b_2\}$ satisfies the conditions of Lemma 6, and so it is 2-connected. Since $(H_k-M'''_{k-1})+\{a_1b_1, a_2b_2\}$ is a spanningsubgraph of H'''_k, H'''_k is 2-connected. Similarly, H'''_k is also Hamiltonian. Let C_k be a Hamilton cycle in H'''_k and let M_k be the maximum matching in H'''_k such that $M_k\subset E(C_k)$ and v_k is not incident with an edge in M_k. Then we replace M_{k-1} by M'''_{k-1} in $\{M_1, \ldots, M_{k-1}, M_k\}$ and still denote M'''_{k-1} by M_{k-1} in $\{M_1, \ldots, M'''_{k-1}, M_k\}$. Therefore, G contains the matching M_k which misses the two vertices u_k and v_k, and does not miss any other vertex.

In each of the above cases, we have shown that the lemma holds. $\square$

Proof of theorem (Conclusion). Let $G' = G-(M_1 \cup \cdots \cup M_r)$, and let $V_1 = \{u_1, v_1, \ldots, u_r, v_r\}$. Form a new graph G^* by adding a new vertex v^* to G' and joining v^* to each vertex of $V(G) - V_1$ by an edge. We next discuss the maximum degree of G^*. It is evident that

$$d_{G^*}(v^*) = n - 2(n - \Delta(G) - 1) = 2\Delta(G) - n + 2.$$

Case (a): $x \in V_1$. Then x is missed by precisely one M_k and is nonadjacent to v^* in G^* and so

$$\begin{aligned} d_{G^*}(x) &= \Delta(G) - (r-1) = \Delta(G) - (n - \Delta(G) - 1) + 1 \\ &= 2\Delta(G) - n + 2. \end{aligned}$$

Case (b): $x \notin V_1$. Then x is adjacent to v^* in G^* and is not missed by any M_k, and so

$$d_{G^*}(x) = \Delta(G) - r + 1 = 2\Delta(G) - n + 2.$$

Thus, $\Delta(G^*) = 2\Delta(G) - n + 2$. By Lemma 2 , there exists an edge colouring φ of G^* that uses $2\Delta(G) - n + 3$ colours. We now form a total colouring π of G that uses $\Delta(G) + 2$ colours as follows:

$$\begin{array}{ll} \pi(e) = \varphi(e) & \text{if } e \in E(G) \cap E(G^*), \\ \pi(v) = \varphi(vv^*) & \text{if } vv^* \in E(G^*), \\ \pi(e) = k & \text{if } e \in M_k, \text{ for } k = 1, \ldots, n - \Delta(G) - 1, \\ \pi(u_k) = \pi(v_k) = k & \text{if } u_k, v_k \in V_1, \text{ for } k = 1, \ldots, n - \Delta(G) - 1. \end{array}$$

It can be verified that π is indeed a total colouring of G. $\square$

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